Supplementary material for LHCb-PAPER-2016-015

1 Details of the matrix element for the decay amplitude

1.1 Helicity formalism and notation

For each two-body decay $A \to B\,C$, a coordinate system is set up in the rest frame of A, with $\hat{z}$ being[1] the direction of quantization for its spin. We denote this coordinate system as $(x_0^{\{A\}}, y_0^{\{A\}}, z_0^{\{A\}})$, where the superscript "$\{A\}$" means "in the rest frame of A", while the subscript "0" means the initial coordinates. For the first particle in the decay chain (Λ_b^0), the choice of these coordinates is arbitrary.[2] However, once defined, these coordinates must be used consistently between all decay sequences described by the matrix element. For subsequent decays, *e.g.* $B \to D\,E$, the choice of these coordinates is already fixed by the transformation from the A to the B rest frames, as discussed below. Helicity is defined as the projection of the spin of the particle onto the direction of its momentum. When the z axis coincides with the particle momentum, we denote its spin projection onto it (*i.e.* the m_z quantum number) as λ. To use the helicity formalism, the initial coordinate system must be rotated to align the z axis with the direction of the momentum of one of the child particles, *e.g.* the B. A generalized rotation operator can be formulated in three-dimensional space, $\mathcal{R}(\alpha, \beta, \gamma)$, that uses Euler angles. Applying this operator results in a sequence of rotations: first by the angle α about the $\hat{z}_0$ axis, followed by the angle β about the rotated $\hat{y}_1$ axis and then finally by the angle γ about the rotated $\hat{z}_2$ axis. We use a subscript denoting the axes, to specify the rotations which have been already performed on the coordinates. The spin eigenstates of particle A, $|J_A, m_A\rangle$, in the $(x_0^{\{A\}}, y_0^{\{A\}}, z_0^{\{A\}})$ coordinate system can be expressed in the basis of its spin eigenstates, $|J_A, m'_A\rangle$, in the rotated $(x_3^{\{A\}}, y_3^{\{A\}}, z_3^{\{A\}})$ coordinate system with the help of Wigner's D-matrices

$$|J_A, m_A\rangle = \sum_{m'_A} D^{J_A}_{m_A, m'_A}(\alpha, \beta, \gamma)^* \, |J_A, m'_A\rangle, \tag{1}$$

where

$$D^J_{m,m'}(\alpha, \beta, \gamma)^* = \langle J, m|\mathcal{R}(\alpha, \beta, \gamma)|J, m'\rangle^* = e^{i\,m\alpha}\; d^J_{m,m'}(\beta)\; e^{i\,m'\gamma}, \tag{2}$$

and where the small-d Wigner matrix contains known functions of β that depend on J, m, m'. To achieve the rotation of the original $\hat{z}_0^{\{A\}}$ axis onto the B momentum ($\vec{p}_B^{\{A\}}$), it is sufficient to rotate by $\alpha = \phi_B^{\{A\}}$, $\beta = \theta_B^{\{A\}}$, where $\phi_B^{\{A\}}$, $\theta_B^{\{A\}}$ are the azimuthal and polar angles of the B momentum vector in the original coordinates *i.e.* $(\hat{x}_0^{\{A\}}, \hat{y}_0^{\{A\}}, \hat{z}_0^{\{A\}})$. This is depicted in Fig. 1, for the case when the quantization axis for the spin of A is its

[1] The "hat" symbol denotes a unit vector in a given direction.

[2] When designing an analysis to be sensitive (or insensitive) to a particular case of polarization, the choice is not arbitrary, but this does not change the fact that one can quantize the Λ_b^0 spin along any well-defined direction. The Λ_b^0 polarization may be different for different choices.

momentum in some other reference frame. Since the third rotation is not necessary, we set $\gamma = 0$.[3] The angle $\theta_B^{\{A\}}$ is usually called "the A helicity angle", thus to simplify the notation we will denote it as θ_A. For compact notation, we will also denote $\phi_B^{\{A\}}$ as ϕ_B. These angles can be determined from[4]

$$\begin{aligned}\phi_B &= \mathrm{atan2}\left(p_B^{\{A\}}{}_y,\, p_B^{\{A\}}{}_x\right)\\ &= \mathrm{atan2}\left(\hat{y}_0^{\{A\}}\cdot\vec{p}_B^{\{A\}},\, \hat{x}_0^{\{A\}}\cdot\vec{p}_B^{\{A\}}\right)\\ &= \mathrm{atan2}\left((\hat{z}_0^{\{A\}}\times\hat{x}_0^{\{A\}})\cdot\vec{p}_B^{\{A\}},\, \hat{x}_0^{\{A\}}\cdot\vec{p}_B^{\{A\}}\right), \qquad (3)\\ \cos\theta_A &= \hat{z}_0^{\{A\}}\cdot\hat{p}_B^{\{A\}}. \qquad (4)\end{aligned}$$

Angular momentum conservation requires $m'_A = m'_B + m'_C = \lambda_B - \lambda_C$ (since $\vec{p}_C^{\{A\}}$ points in the opposite direction to $\hat{z}_3^{\{A\}}$, $m'_C = -\lambda_C$). Each two-body decay contributes a multiplicative term to the matrix element

$$\mathcal{H}^{A\to B\,C}_{\lambda_B,\lambda_C}\, D^{J_A}_{m_A,\,\lambda_B-\lambda_C}(\phi_B,\theta_A,0)^*. \qquad (5)$$

The helicity couplings $\mathcal{H}^{A\to B\,C}_{\lambda_B,\lambda_C}$ are complex constants. Their products from subsequent decays are to be determined by the fit to the data (they represent the decay dynamics). If the decay is strong or electromagnetic, it conserves parity which reduces the number of independent helicity couplings via the relation

$$\mathcal{H}^{A\to B\,C}_{-\lambda_B,-\lambda_C} = P_A\,P_B\,P_C\,(-1)^{J_B+J_C-J_A}\,\mathcal{H}^{A\to B\,C}_{\lambda_B,\lambda_C}, \qquad (6)$$

where P stands for the intrinsic parity of a particle.

After multiplying terms given by Eq. (5) for all decays in the decay sequence, they must be summed up coherently over the helicity states of intermediate particles, and incoherently over the helicity states of the initial and final-state particles. Possible helicity values of B and C particles are constrained by $|\lambda_B| \le J_B$, $|\lambda_C| \le J_C$ and $|\lambda_B - \lambda_C| \le J_A$.

When dealing with the subsequent decay of the child, $B \to D\,E$, four-vectors of all particles must be first Lorentz boosted to the rest frame of B, along the $\vec{p}_B^{\{A\}}$ *i.e.* $\hat{z}_3^{\{A\}}$ direction (this is the z axis in the rest frame of A after the Euler rotations; we use the subscript "3" for the number of rotations performed on the coordinates, because of the three Euler angles, however, since we use the $\gamma = 0$ convention these coordinates are the same as after the first two rotations). This is visualized in Fig. 1, with $B \to D\,E$ particle labels replaced by $A \to B\,C$ labels. This transformation does not change vectors that are perpendicular to the boost direction. The transformed coordinates become the initial

[3] An alternate convention is to set $\gamma = -\alpha$. The two conventions lead to equivalent formulae.

[4] The function $\mathrm{atan2}(x, y)$ is the $\tan^{-1}(y/x)$ function with two arguments. The purpose of using two arguments instead of one is to gather information on the signs of the inputs in order to return the appropriate quadrant of the computed angle.

coordinate system quantizing the spin of B in its rest frame,

$$\begin{aligned} \hat{x}_0^{\{B\}} &= \hat{x}_3^{\{A\}}, \\ \hat{y}_0^{\{B\}} &= \hat{y}_3^{\{A\}}, \\ \hat{z}_0^{\{B\}} &= \hat{z}_3^{\{A\}}. \end{aligned} \tag{7}$$

The processes of rotation and subsequent boosting can be repeated until the final-state particles are reached. In practice, there are two equivalent ways to determine the $\hat{z}_0^{\{B\}}$ direction. Using Eq. (7) we can set it to the direction of the B momentum in the A rest frame

$$\hat{z}_0^{\{B\}} = \hat{z}_3^{\{A\}} = \hat{p}_B^{\{A\}}. \tag{8}$$

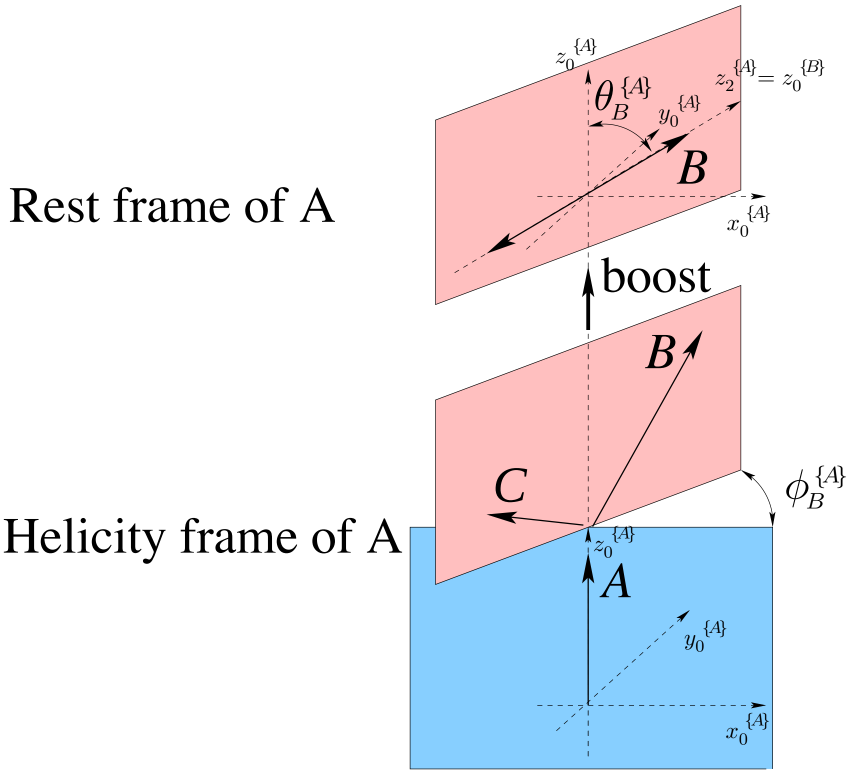

Figure 1: Coordinate axes for the spin quantization of particle A (bottom part), chosen to be the helicity frame of A ($\hat{z}_0 || \vec{p}_A$ in the rest frame of its parent particle or in the laboratory frame), together with the polar ($\theta_B^{\{A\}}$) and azimuthal ($\phi_B^{\{A\}}$) angles of the momentum of its child B in the A rest frame (top part). Notice that the directions of these coordinate axes, denoted as $\hat{x}_0^{\{A\}}$, $\hat{y}_0^{\{A\}}$, and $\hat{z}_0^{\{A\}}$, do not change when boosting from the helicity frame of A to its rest frame. After the Euler rotation $\mathcal{R}(\alpha = \phi_B^{\{A\}}, \beta = \theta_B^{\{A\}}, \gamma = 0)$ (see the text), the rotated z axis, $\hat{z}_2^{\{A\}}$, is aligned with the B momentum; thus the rotated coordinates become the helicity frame of B. If B has a sequential decay, then the same boost-rotation process is repeated to define the helicity frame for its decay products.

Alternatively, we can make use of the fact that B and C are back-to-back in the rest frame of A, $\vec{p}_C^{\,\{A\}} = -\vec{p}_B^{\,\{A\}}$. Since the momentum of C is antiparallel to the boost direction from the A to B rest frames, the C momentum in the B rest frame will be different, but it will still be antiparallel to this boost direction

$$\hat{z}_0^{\{B\}} = -\hat{p}_C^{\{B\}}. \tag{9}$$

To determine $\hat{x}_0^{\{B\}}$ from Eq. (7), we need to find $\hat{x}_3^{\{A\}}$. After the first rotation by ϕ_B about $\hat{z}_0^{\{A\}}$, the $\hat{x}_1^{\{A\}}$ axis is along the component of $\vec{p}_B^{\,\{A\}}$ which is perpendicular to the $\hat{z}_0^{\{A\}}$ axis

$$\begin{aligned} \vec{a}_{B\perp z_0}^{\,\{A\}} &\equiv (\vec{p}_B^{\,\{A\}})_{\perp \hat{z}_0^{\{A\}}} = \vec{p}_B^{\,\{A\}} - (\vec{p}_B^{\,\{A\}})_{||\hat{z}_0^{\{A\}}}, \\ &= \vec{p}_B^{\,\{A\}} - (\vec{p}_B^{\,\{A\}} \cdot \hat{z}_0^{\{A\}})\, \hat{z}_0^{\{A\}}, \\ \hat{x}_1^{\{A\}} &= \hat{a}_{B\perp z_0}^{\{A\}} = \frac{\vec{a}_{B\perp z_0}^{\,\{A\}}}{|\,\vec{a}_{B\perp z_0}^{\,\{A\}}\,|}. \end{aligned} \tag{10}$$

After the second rotation by θ_A about $\hat{y}_1^{\{A\}}$, $\hat{z}_2^{\{A\}} \equiv \hat{z}_3^{\{A\}} = \hat{p}_B^{\{A\}}$, and $\hat{x}_2^{\{A\}} = \hat{x}_3^{\{A\}}$ is antiparallel to the component of the $\hat{z}_0^{\{A\}}$ vector that is perpendicular to the new z axis *i.e.* $\hat{p}_B^{\{A\}}$. Thus

$$\begin{aligned} \vec{a}_{z_0\perp B}^{\,\{A\}} &\equiv (\hat{z}_0^{\{A\}})_{\perp \vec{p}_B^{\{A\}}} = \hat{z}_0^{\{A\}} - (\hat{z}_0^{\{A\}} \cdot \hat{p}_B^{\{A\}})\, \hat{p}_B^{\{A\}}, \\ \hat{x}_0^{\{B\}} &= \hat{x}_3^{\{A\}} = -\, \hat{a}_{z_0\perp B}^{\{A\}} = -\, \frac{\vec{a}_{z_0\perp B}^{\,\{A\}}}{|\,\vec{a}_{z_0\perp B}^{\,\{A\}}\,|}. \end{aligned} \tag{11}$$

Then we obtain $\hat{y}_0^{\{B\}} = \hat{z}_0^{\{B\}} \times \hat{x}_0^{\{B\}}$.

If C also decays, $C \to F\,G$, then the coordinates for the quantization of C spin in the C rest frame are defined by

$$\hat{z}_0^{\{C\}} = -\hat{z}_3^{\{A\}} = \hat{p}_C^{\{A\}} = -\hat{p}_B^{\{C\}}, \tag{12}$$

$$\hat{x}_0^{\{C\}} = \hat{x}_3^{\{A\}} = -\, \hat{a}_{z_0\perp B}^{\{A\}} = +\hat{a}_{z_0\perp C}^{\{A\}}, \tag{13}$$

$$\hat{y}_0^{\{C\}} = \hat{z}_0^{\{C\}} \times \hat{x}_0^{\{C\}}, \tag{14}$$

i.e. the z axis is reflected compared to the system used for the decay of particle B (it must point in the direction of C momentum in the A rest frame), but the x axis is kept the same, since we chose particle B for the rotation used in Eq. (5).

1.2 Matrix element for the Z_c^- decay chain

The decay matrix elements for the two interfering decay chains, $\Lambda_b^0 \to J/\psi\, N^*$, $N^* \to p\pi^-$ and $\Lambda_b^0 \to P_c^+\pi^-$, $P_c^+ \to J/\psi\, p$ with $J/\psi \to \mu^+\mu^-$ in both cases, are identical to those used in the $\Lambda_b^0 \to J/\psi\, pK^-$ analysis [1], with K^- and Λ^* replaced by π^- and N^*. We now turn to the discussion of the additional interfering decay chain, $\Lambda_b^0 \to Z_{cf}p$, $Z_{cf} \to \psi\pi^-$ decays (denoting J/ψ as ψ), in which we allow more than one tetraquark state, $f = 1, 2, \ldots$. Superscripts containing the Z_c decay chain name without curly brackets, *e.g.* ϕ^{Z_c}, will denote quantities belonging to this decay chain and should not be confused with the superscript "$\{Z_c\}$" denoting the Z_c rest frame, *e.g.* $\phi^{\{Z_c\}}$. With only a few exceptions, we omit the N^* decay chain label. The angular calculations for the Z_c^- decay chain are analogous to that for P_c^+ by interchange of p and π^-, except for the angles to align the proton helicity.

The weak decay $\Lambda_b^0 \to Z_{cf} p$ is described by the term,

$$\mathcal{H}^{\Lambda_b^0 \to Z_{cf} p}_{\lambda_{Z_c}, \lambda_p^{Z_c}} \; D^{\frac{1}{2}}_{\lambda_{\Lambda_b^0}, \lambda_{Z_c} - \lambda_p^{Z_c}}(\phi_{Z_c}, \theta^{Z_c}_{\Lambda_b^0}, 0)^*, \tag{15}$$

where $\mathcal{H}^{\Lambda_b^0 \to Z_{cf} p}_{\lambda_{Z_c}, \lambda_p^{Z_c}}$ are resonance (*i.e.* f) dependent helicity couplings. As for $|\lambda_{Z_c} - \lambda_p^{Z_c}| \leq \frac{1}{2}$, there are two different helicity couplings for $J_{Z_c} = 0$, and four for higher spin. The above mentioned ϕ_{Z_c}, $\theta^{Z_c}_{\Lambda_b^0}$ symbols refer to the azimuthal and polar angles of Z_c in the Λ_b^0 rest frame (see Fig. 2).

With the direction of Λ_b^0 in the lab frame $\hat{p}^{\{\text{lab}\}}_{\Lambda_b^0}$, and the direction of Z_c in the Λ_b^0 rest frame, the Λ_b^0 helicity angle in the Z_c decay chain can be calculated as,

$$\cos\theta^{Z_c}_{\Lambda_b^0} = \hat{p}^{\{\text{lab}\}}_{\Lambda_b^0} \cdot \hat{p}^{\{\Lambda_b^0\}}_{Z_c}. \tag{16}$$

The ϕ_{Z_c} angle cannot be set to zero, since we have already defined the $\hat{x}_0^{\{\Lambda_b^0\}}$ axis in

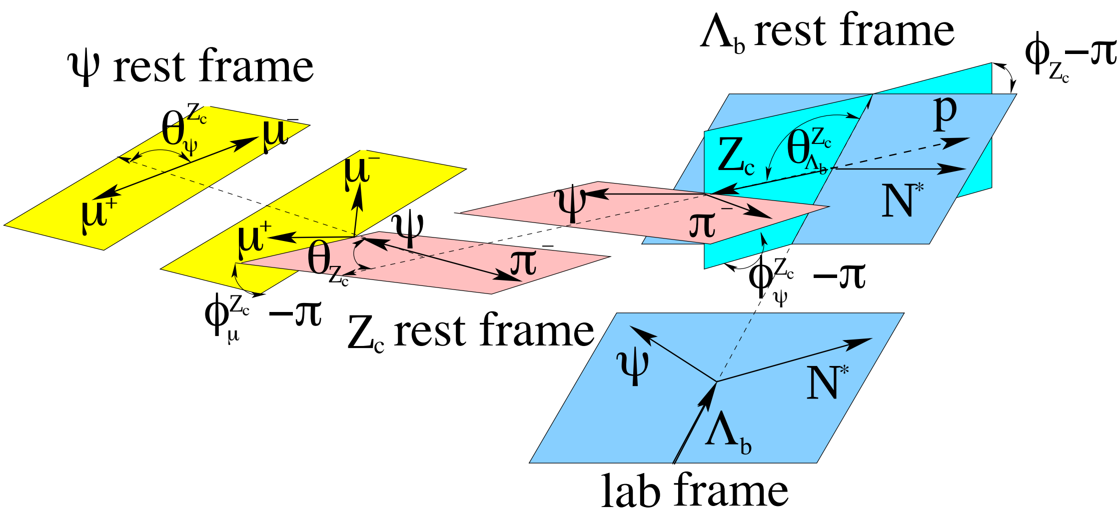

Figure 2: Definition of the decay angles in the Z_c^- decay chain.

the Λ_b^0 rest frame by the $\phi_{N^*} = 0$ convention:

$$\begin{aligned}
\vec{a}_{N^*\perp z_0}^{\{\Lambda_b^0\}} &= \vec{p}_{N^*}^{\{\Lambda_b^0\}} - (\vec{p}_{N^*}^{\{\Lambda_b^0\}} \cdot \hat{p}_{\Lambda_b^0}^{\{\text{lab}\}})\, \hat{p}_{\Lambda_b^0}^{\{\text{lab}\}}, \\
\hat{x}_0^{\{\Lambda_b^0\}} &= \frac{\vec{a}_{N^*\perp z_0}^{\{\Lambda_b^0\}}}{|\, \vec{a}_{N^*\perp z_0}^{\{\Lambda_b^0\}} \,|}.
\end{aligned} \tag{17}$$

The ϕ_{Z_c} angle can be determined in the Λ_b^0 rest frame from

$$\phi_{Z_c} = \text{atan2}\left((\hat{p}_{\Lambda_b^0}^{\{\text{lab}\}} \times \hat{x}_0^{\{\Lambda_b^0\}}) \cdot \hat{p}_{Z_c}^{\{\Lambda_b^0\}},\ \hat{x}_0^{\{\Lambda_b^0\}} \cdot \hat{p}_{Z_c}^{\{\Lambda_b^0\}} \right). \tag{18}$$

The strong decay $Z_{cf} \to \psi\pi^-$ is described by a term

$$\mathcal{H}_{\lambda_\psi^{Z_c}}^{Z_{cf}\to\psi\pi}\ D_{\lambda_{Z_c},\lambda_\psi^{Z_c}}^{J_{Z_{cf}}}(\phi_\psi^{Z_c}, \theta_{Z_c}, 0)^*\ R_{Z_{cf}}(M_{\psi\pi}), \tag{19}$$

where $\phi_\psi^{Z_c}, \theta_{Z_c}$ are the azimuthal and polar angles of the ψ in the Z_c rest frame (see Fig. 2). The $\hat{z}_0^{\{Z_c\}}$ direction is defined by the boost direction from the Λ_b^0 rest frame, which coincides with the $-\vec{p}_p^{\{Z_c\}}$ direction. This leads to

$$\cos\theta_{Z_c} = -\hat{p}_p^{\{Z_c\}} \cdot \hat{p}_\psi^{\{Z_c\}}. \tag{20}$$

The azimuthal angle of the ψ can now be determined in the Z_c rest frame (see Fig. 2) from

$$\phi_\psi^{Z_c} = \text{atan2}\left(-(\hat{p}_p^{\{Z_c\}} \times \hat{x}_0^{\{Z_c\}}) \cdot \hat{p}_\psi^{\{Z_c\}},\ \hat{x}_0^{\{Z_c\}} \cdot \hat{p}_\psi^{\{Z_c\}} \right). \tag{21}$$

The $\hat{x}_0^{\{Z_c\}}$ direction is defined by the convention that we used in the Λ_b^0 rest frame. Thus, we have

$$\begin{aligned}
\vec{a}_{z_0\perp Z_c}^{\{\Lambda_b^0\}} &= \hat{p}_{\Lambda_b^0}^{\{\text{lab}\}} - (\hat{p}_{\Lambda_b^0}^{\{\text{lab}\}} \cdot \hat{p}_{Z_c}^{\{\Lambda_b^0\}})\, \hat{p}_{Z_c}^{\{\Lambda_b^0\}}, \\
\hat{x}_0^{\{Z_c\}} &= -\frac{\vec{a}_{z_0\perp Z_c}^{\{\Lambda_b^0\}}}{|\, \vec{a}_{z_0\perp Z_c}^{\{\Lambda_b^0\}} \,|}.
\end{aligned} \tag{22}$$

Again, the ψ and p helicities are labeled as $\lambda_\psi^{Z_c}$ and $\lambda_p^{Z_c}$, with the Z_c superscript to make it clear that the spin quantization axes are different than in the N^* decay chain. Since the ψ is an intermediate particle, this has no consequences after we sum (coherently) over $\lambda_\psi^{Z_c} = -1, 0, +1$. The proton, however, is a final-state particle. Before the Z_c terms in the matrix element can be added coherently to the N^* terms, the $\lambda_p^{Z_c}$ states must be rotated to λ_p states (defined in the N^* decay chain). The proton helicity axes are different, since the proton comes from a decay of different particles in the two decay sequences, the N^* and Λ_b^0. The quantization axes are along the proton direction in the N^* and the Λ_b^0 rest frames, thus antiparallel to the particles recoiling against the proton: the π^- and Z_c, respectively. These directions are preserved when boosting to the proton rest frame. Thus, the polar angle between the two proton quantization axes ($\theta_p^{Z_c}$) can be determined from the opening angle between the π^- and Z_c mesons in the p rest frame,

$$\cos\theta_p^{Z_c} = \hat{p}_\pi^{\{p\}} \cdot \hat{p}_{Z_c}^{\{p\}}. \tag{23}$$

The dot product above must be calculated by operating on the $\vec{p}_\pi^{\{p\}}$ and $\vec{p}_{Z_c}^{\{p\}}$ vectors in the proton rest frame obtained by the same sequence of boost transformations, either according to the N^* or Z_c decay chains, or even by a direct boost transformation from the lab frame.

Unlike in the P_c decay chain, the azimuthal angle ($\alpha_p^{Z_c}$) aligning the two proton helicity frames is not zero. The angle can be determined from

$$\alpha_p^{Z_c} = \operatorname{atan2}\left((\hat{z}_0^{\{p\}\ \Lambda_b^0} \times \hat{x}_0^{\{p\}\ \Lambda_b^0}) \cdot \hat{x}_0^{\{p\}\ N^*}, \hat{x}_0^{\{p\}\ \Lambda_b^0} \cdot \hat{x}_0^{\{p\}\ N^*} \right), \tag{24}$$

where all vectors are in the p rest frame, $\hat{x}_0^{\{p\}\ N^*}$ is the direction of the x axis when boosting from the N^* rest frame, $\hat{x}_0^{\{p\}\ \Lambda_b^0}$ and $\hat{z}_0^{\{p\}\ \Lambda_b^0}$ are the directions of the x and z axes when boosting from the Λ_b^0 rest frame. From Eq. (9), $\hat{z}_0^{\{p\}\ \Lambda_b^0} = -\hat{p}_{Z_c}^{\{p\}}$. Direction of $\hat{x}_0^{\{p\}\ \Lambda_b^0}$ is given by Eq.(11) with $\hat{z}_0^{\{\Lambda_b^0\}} = \hat{p}_{\Lambda_b^0}^{\{\text{lab}\}}$

$$\begin{aligned}
\vec{a}_{z_0\perp Z_c}^{\{\Lambda_b^0\}} &= \hat{p}_{\Lambda_b^0}^{\{\text{lab}\}} - (\hat{p}_{\Lambda_b^0}^{\{\text{lab}\}} \cdot \hat{p}_{Z_c}^{\{\Lambda_b^0\}})\, \hat{p}_{Z_c}^{\{\Lambda_b^0\}}, \\
\hat{x}_0^{\{p\}\ \Lambda_b^0} &= \frac{\vec{a}_{z_0\perp Z_c}^{\{\Lambda_b^0\}}}{|\, \vec{a}_{z_0\perp Z_c}^{\{\Lambda_b^0\}}\,|}.
\end{aligned} \tag{25}$$

Therefore, the relation between λ_p and $\lambda_p^{Z_c}$ states is

$$|\lambda_p\rangle = \sum_{\lambda_p^{Z_c}} D_{\lambda_p^{Z_c},\,\lambda_p}^{\ J_p}(\alpha_p^{Z_c}, \theta_p^{Z_c}, 0)^* |\lambda_p^{Z_c}\rangle = \sum_{\lambda_p^{Z_c}} e^{i\lambda_p^{Z_c}\alpha_p^{Z_c}} d_{\lambda_p^{Z_c},\,\lambda_p}^{\ J_p}(\theta_p^{Z_c}) |\lambda_p^{Z_c}\rangle. \tag{26}$$

Thus, the term given by Eq. (19) must be preceded by

$$\sum_{\lambda_p^{Z_c}=\pm\frac{1}{2}} e^{i\lambda_p^{Z_c}\alpha_p^{Z_c}} d_{\lambda_p^{Z_c},\,\lambda_p}^{\ J_p}(\theta_p^{Z_c}). \tag{27}$$

Parity conservation in $Z_{cf} \to \psi\pi^-$ decays leads to the following relation

$$\begin{aligned}
\mathcal{H}_{-\lambda_\psi^{Z_c}}^{Z_{cf}\to\psi\pi} &= P_\psi\, P_\pi\, P_{Z_{cf}}\, (-1)^{J_\psi + J_K - J_{Z_{cf}}}\, \mathcal{H}_{\lambda_\psi^{Z_c}}^{Z_{cf}\to\psi\pi} \\
&= P_{Z_{cf}}\, (-1)^{1-J_{Z_{cf}}}\, \mathcal{H}_{\lambda_\psi^{Z_c}}^{Z_{cf}\to\psi\pi},
\end{aligned} \tag{28}$$

where $P_{Z_{cf}}$ is the parity of the Z_{cf} state. Then the number of independent helicity couplings to be determined from the data is reduced to two for $J_{Z_{cf}} \geq 1$ and remains equal to unity for $J_{Z_{cf}} = 0$. Since the helicity couplings enter the matrix element formula as a product, $\mathcal{H}^{\Lambda_b^0\to Z_{cf}p}_{\lambda_{Z_c},\lambda_p^{Z_c}}\,\mathcal{H}^{Z_{cf}\to\psi\pi}_{\lambda_\psi^{Z_c}}$, the relative magnitude and phase of these two sets must be fixed by a convention. For example, $\mathcal{H}^{Z_{cf}\to\psi\pi}_{\lambda_\psi^{Z_c}=0}$ can be set to $(1,0)$ for every Z_{cf} resonance, in which case $\mathcal{H}^{Z_{cf}\to\psi\pi}_{\lambda_\psi^{Z_c}=1}$ develops a meaning of the complex ratio of $\mathcal{H}^{Z_{cf}\to\psi\pi}_{\lambda_\psi^{Z_c}=1}/\mathcal{H}^{Z_{cf}\to\psi\pi}_{\lambda_\psi^{Z_c}=0}$, while all $\mathcal{H}^{\Lambda_b^0\to Z_{cf}p}_{\lambda_{Z_c},\lambda_p^{Z_c}}$ couplings should have both real and imaginary parts free in the fit.

The term $R_{Z_{cf}}(M_{\psi\pi})$ in Eq. (19) describes the $\psi\pi$ invariant mass distribution of the Z_{cf} resonance. Angular momentum conservation restricts $\max(J_{Z_{cf}}-1,0)\le L^{Z_{cf}}_{\Lambda_b^0}\le J_{Z_{cf}}+1$ in $\Lambda_b^0\to Z_{cf}p$ decays. Angular momentum conservation also imposes $\max(|J_{Z_{cf}}-1|\,,\,0)\le L_{Z_{cf}}\le J_{Z_{cf}}+1$, which is further restricted by the parity conservation in the Z_{cf} decays, $P_{Z_{cf}}=(-1)^{L_{Z_{cf}}}$. We assume the minimal values of $L^{Z_{cf}}_{\Lambda_b^0}$ and of $L_{Z_{cf}}$ in $R_{Z_{cf}}(m_{\psi\pi})$.

The electromagnetic decay $\psi\to\mu^+\mu^-$ in the Z_c decay chain contributes a term

$$D^{\,1}_{\lambda_\psi^{Z_c},\,\Delta\lambda_\mu^{Z_c}}(\phi_\mu^{Z_c},\theta_\psi^{Z_c},0)^*. \tag{29}$$

The azimuthal and polar angle of the muon in the ψ rest frame, $\phi_\mu^{Z_c}$, $\theta_\psi^{Z_c}$, are different from ϕ_μ, θ_ψ introduced in the N^* decay chain. The ψ helicity axis is along the boost direction from the Z_c to the ψ rest frames, which is given by

$$\hat{z}_0^{\{\psi\}\ Z_c} = -\,\hat{p}_\pi^{\{\psi\}}, \tag{30}$$

and so

$$\cos\theta_\psi^{Z_c} = -\hat{p}_\pi^{\{\psi\}}\cdot\hat{p}_\mu^{\{\psi\}}. \tag{31}$$

The x axis is inherited from the Z_c rest frame (Eq. (11)),

$$\begin{aligned}\vec{a}_{z_0\perp\psi}^{\{Z_c\}} &= -\vec{p}_p^{\{Z_c\}} + (\vec{p}_p^{\{Z_c\}}\cdot\hat{p}_\psi^{\{Z_c\}})\,\hat{p}_\psi^{\{Z_c\}}\\ \hat{x}_0^{\{\psi\}\ Z_c} = \hat{x}_3^{\{Z_c\}} &= -\,\frac{\vec{a}_{z_0\perp\psi}^{\{Z_c\}}}{|\,\vec{a}_{z_0\perp\psi}^{\{Z_c\}}\,|},\end{aligned} \tag{32}$$

which leads to

$$\phi_\mu^{Z_c} = \mathrm{atan2}\left(-(\hat{p}_\pi^{\{\psi\}}\times\hat{x}_0^{\{\psi\}\ Z_c})\cdot\hat{p}_\mu^{\{\psi\}},\ \hat{x}_0^{\{\psi\}\ Z_c}\cdot\hat{p}_\mu^{\{\psi\}}\right). \tag{33}$$

The azimuthal angle $\alpha_\mu^{Z_c}$ is defined by

$$\alpha_\mu^{Z_c} = \mathrm{atan2}\left((\hat{z}_3^{\{\psi\}\ Z_c}\times\hat{x}_3^{\{\psi\}\ Z_c})\cdot\hat{x}_3^{\{\psi\}\ N^*},\ \hat{x}_3^{\{\psi\}\ Z_c}\cdot\hat{x}_3^{\{\psi\}\ N^*}\right), \tag{34}$$

where $\hat{z}_3^{\{\psi\}\ Z_c}=\hat{p}_\mu^{\{\psi\}\ Z_c}$, and from Eq. (11)

$$\hat{x}_3^{\{\psi\}\ Z_c} = -\,\hat{a}_{z_0\perp\mu}^{\{\psi\}\ \ Z_c}, \tag{35}$$

$$\vec{a}_{z_0\perp\mu}^{\{\psi\}\ \ Z_c} = -\,\hat{p}_\pi^{\{\psi\}} + (\hat{p}_\pi^{\{\psi\}}\cdot\hat{p}_\mu^{\{\psi\}})\,\hat{p}_\mu^{\{\psi\}}, \tag{36}$$

as well as

$$\hat{x}_3^{\{\psi\}\ N^*} = -\,\hat{a}_{z_0\perp\mu}^{\{\psi\}\ \ N^*}, \tag{37}$$

$$\vec{a}_{z_0\perp\mu}^{\{\psi\}\ \ N^*} = -\,\hat{p}_{N^*}^{\{\psi\}} + (\hat{p}_{N^*}^{\{\psi\}}\cdot\hat{p}_{\mu}^{\{\psi\}})\,\hat{p}_{\mu}^{\{\psi\}}. \tag{38}$$

Collecting terms from the three subsequent decays in the Z_c chain together,

$$\begin{aligned}\mathcal{M}^{Z_c}_{\lambda_{\Lambda_b^0},\lambda_p^{Z_c},\Delta\lambda_\mu^{Z_c}} =& e^{i\,\lambda_{\Lambda_b^0}\phi_{Z_c}}\sum_f R_{Z_{cf}}(M_{\psi\pi})\sum_{\lambda_\psi^{Z_c}} e^{i\,\lambda_\psi^{Z_c}\phi_\mu^{Z_c}}\ d^{\,1}_{\lambda_\psi^{Z_c},\,\Delta\lambda_\mu}(\theta_\psi^{Z_c}) \\ &\times\sum_{\lambda_{Z_c}}\mathcal{H}^{\Lambda_b^0\to Z_{cf}p}_{\lambda_{Z_c},\,\lambda_p^{Z_c}}\ e^{i\,\lambda_{Z_c}\phi_\psi^{Z_c}}\ d^{\,\frac{1}{2}}_{\lambda_{\Lambda_b^0},\,\lambda_{Z_c}-\lambda_p^{Z_c}}(\theta_{\Lambda_b^0}^{Z_c})\mathcal{H}^{Z_{cf}\to\psi\pi}_{\lambda_\psi^{Z_c}}\ d^{\,J_{Z_{cf}}}_{\lambda_{Z_c},\,\lambda_\psi^{Z_c}}(\theta_{Z_c}),\end{aligned} \tag{39}$$

and adding them coherently to the N^* and the P_c matrix elements, via appropriate relations of $|\lambda_p\rangle|\lambda_{\mu^+}\rangle|\lambda_{\mu^-}\rangle$ to $|\lambda_p^{P_c}\rangle|\lambda_{\mu^+}^{P_c}\rangle|\lambda_{\mu^-}^{P_c}\rangle$ and $|\lambda_p^{Z_c}\rangle|\lambda_{\mu^+}^{Z_c}\rangle|\lambda_{\mu^-}^{Z_c}\rangle$ states as discussed above, leads to the final matrix element squared

$$\begin{aligned}|\mathcal{M}|^2 = \sum_{\lambda_{\Lambda_b^0}=\pm\frac{1}{2}}\ \sum_{\lambda_p=\pm\frac{1}{2}}\ \sum_{\Delta\lambda_\mu=\pm 1}\Bigg|&\mathcal{M}^{N^*}_{\lambda_{\Lambda_b^0},\,\lambda_p,\,\Delta\lambda_\mu} + e^{i\,\Delta\lambda_\mu\alpha_\mu}\sum_{\lambda_p^{P_c}} d^{\,\frac{1}{2}}_{\lambda_p^{P_c},\,\lambda_p}(\theta_p)\,\mathcal{M}^{P_c}_{\lambda_{\Lambda_b^0},\,\lambda_p^{P_c},\,\Delta\lambda_\mu} \\ &+ e^{i\,\Delta\lambda_\mu\alpha_\mu^{Z_c}}\sum_{\lambda_p^{Z_c}} e^{i\lambda_p^{Z_c}\alpha_p^{Z_c}} d^{\,\frac{1}{2}}_{\lambda_p^{Z_c},\,\lambda_p}(\theta_p^{Z_c})\,\mathcal{M}^{Z_c}_{\lambda_{\Lambda_b^0},\,\lambda_p^{Z_c},\,\Delta\lambda_\mu}\Bigg|^2.\end{aligned} \tag{40}$$

Assuming approximate CP symmetry, the helicity couplings for Λ_b^0 and $\overline{\Lambda}_b^0$ can be made equal, but the calculation of the angles requires some care, since parity (P) conservation does not change polar (*i.e.* helicity) angles, but does change azimuthal angles. Thus, not only must $\vec{p}_{\mu^+}$ be used instead of $\vec{p}_{\mu^-}$ for $\overline{\Lambda}_b^0$ candidates (with π^+ and $\bar{p}$ in the final state) in Eqs. (31), (33), and (34), but also all azimuthal angles must be reflected before entering the matrix element formula: $\phi_{Z_c}\to-\phi_{Z_c}$, $\phi_\psi^{Z_c}\to-\phi_\psi^{Z_c}$, $\phi_\mu^{Z_c}\to-\phi_\mu^{Z_c}$, $\alpha_p^{Z_c}\to-\alpha_p^{Z_c}$ and $\alpha_\mu^{Z_c}\to-\alpha_\mu^{Z_c}$ [2].

References

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